

Revisiting the quantum Szilard engine with fully quantum considerations

Hai Li^{a,b}, Jian Zou^{a,*}, Jun-Gang Li^a, Bin Shao^a, Lian-Ao Wu^c

^a*School of Physics, Beijing Institute of Technology, Beijing 100081, China*

^b*School of Information and Electronics Engineering, Shandong Institute of Business and Technology, Yantai 264000, China*

^c*Department of Theoretical Physics and History of Science, The Basque Country University (EHU/UPV), P. O. Box 644, ES-48080 Bilbao, Spain and IKERBASQUE, Basque Foundation for Science, ES-48011 Bilbao, Spain*

Abstract

By considering level shifting during the insertion process we revisit the quantum Szilard engine (QSZE) with fully quantum consideration. We derive the general expressions of the heat absorbed from thermal bath and the total work done to the environment by the system in a cycle with two different cyclic strategies. We find that only the quantum information contributes to the absorbed heat, and the classical information acts like a feedback controller and has no direct effect on the absorbed heat. This is the first demonstration of the different effects of quantum information and classical information for extracting heat from the bath in the QSZE. Moreover, when the well width $L \rightarrow \infty$ or the temperature of the bath $T \rightarrow \infty$ the QSZE reduces to the classical Szilard engine (CSZE), and the total work satisfies the relation $W_{\text{tot}} = k_B T \ln 2$ as obtained by Sang Wook Kim et al. [Phys. Rev. Lett. 106, 070401 (2011)] for one particle case.

Keywords: Quantum Szilard engine, Energy level shifts, Measurement, Quantum information

1. Introduction

Maxwell's Demon could separate hot atoms from cold, and therefore could obtain work from a single heat bath. This seemed to violate the second law of thermodynamics [1, 2] and led to discussions and confusions until 1929, when Szilard devised his "Szilard Engine" (SZE) [3]. The SZE could extract work from a bath using classical information (acquired by measurement of the atom) and establish the connection between work and entropy to reassure the validation of the second law of thermodynamics, as illustrated in Fig. 1. Later, Landauer [4] and Bennett [5] completely analyzed the SZE, and showed that the erasure or reset of the Demon memory costs at least the energy of $k_B T \ln 2$ associated with the entropy decrease of the engine. It was conjectured [6–8] that there exists a general equivalence relation between information and work; namely, that by having any information J about the state of a physical system, it is possible, by allowing the system to relax to its maximum-entropy state, to convert into mechanical work an amount of heat $W = k_B T J$ without any entropy increase in the environment. Moreover, there are many works on the relationship between information and work, and some significant results have been obtained [9–11].

Research interests in the SZE have recently been revived in various theoretical contexts [12–16] and experimental implementations [17–19]. However, early literatures hardly paid attention to fully quantum analysis except for those via measurement process [20, 21]. Sang Wook Kim noticed that work is required in the process of insertion for a quantum Szilard engine (QSZE) [22]. This makes the engine substantially different from its classical counterpart, classical SZE (CSZE). It has been shown in Ref. [22] that work in the insertion, expansion, removal processes, and the entire cycle are $W_{\text{ins}} = -\Delta + k_B T \ln 2$, $W_{\text{exp}} = \Delta$, $W_{\text{rem}} = 0$ and $W_{\text{tot}} = k_B T \ln 2$, respectively, when the insertion process is performed isothermally. Here, $\Delta = \ln[\frac{z(L)}{Z(L/2)}]$, $z(l) = \sum_{n=1}^{\infty} e^{-\beta E_n(l)}$ and $E_n(l) = \frac{\hbar^2 n^2}{8ml^2}$ ($n = 1, 2, 3, \dots$) with \hbar and m being Planck's

*Corresponding author

Email address: zoujian@bit.edu.cn (Jian Zou)

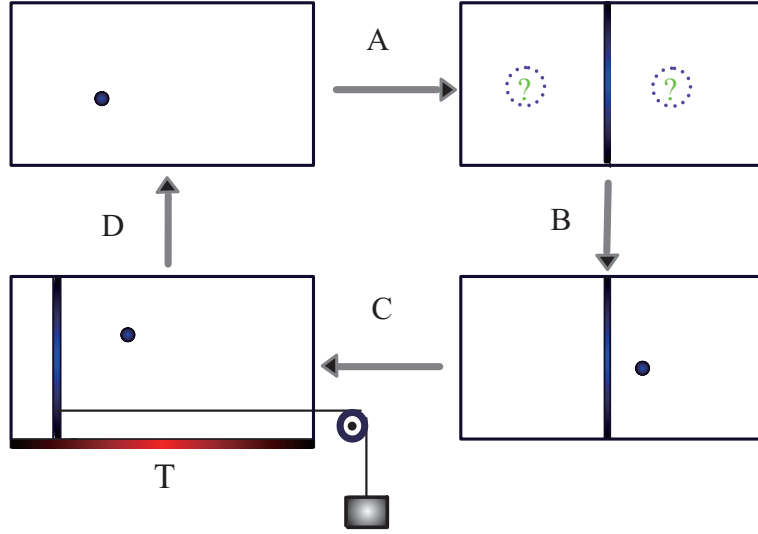


Figure 1: (Color online) Schematic diagram of the thermodynamic processes of the CSZE. Initially a single particle is prepared in an isolated box. (A) The box is divided into two equal subspaces by a wall inserted at the center of box. The dotted circles indicate that before the measurement which subspace the particle locates in is unconfirmed. (B) The particle is found on one side after the measurement. (C) A load is attached to the wall, and the particle absorbs heat and does work via an isothermal expansion at a constant temperature T . (D) To remove the wall which stops at the left end of box, the box returns to the initial situation.

constant and the mass of the particle. In the QSZE, the insertion of the wall is characterized by increase of the height of potential barrier. Energy levels in the box vary with the boundary conditions, contributing to the quantum thermodynamic work and the system's internal energy. Both the position of the insertion and the rate of increase of the potential barrier height influence the level shifts. The faster the height of the potential barrier increases, the greater the change of internal energy of the system and the energy becomes infinite when the height tends to infinity instantaneously, i.e., the insertion is carried out instantaneously, as discussed in [23]. If the system is initially in the ground state and the insertion is performed adiabatically with the barrier being not at the center of the box, the particle will end up definitely in the larger part of the box [23] which is different from the classical situation. In the case of isothermal insertion the effect of energy level shifts is concealed by the heat exchange. So in order to demonstrate the quantum effects of the QSZE completely, it is necessary to consider the adiabatic insertion. In this paper we assume that the insertion is performed adiabatically and analyze the cycle of the QSZE with fully quantum considerations. It is interesting to note that energy level shifts caused by the boundary conditions during the insertion process play a significant role in extracting work or absorbing heat in the QSZE.

In this paper, we revisit the QSZE with fully quantum-mechanical consideration. We consider a single particle in one-dimensional infinite square well, and devise two different cyclic strategies. For these two cyclic strategies we are able to know explicitly the quantities of work done by the system, heat transferred, and the change in internal energy in each step. In this way we can derive the general expressions of heat transferred from the bath and the total work done by the system. We find that the quantum information plays a decisive role in the whole cycle and is associated with heat absorbed from the bath and work done by the system. However, the classical information of the particle being located at seems to behave like a feedback controller, and has no direct effect on heat absorption. This is the first demonstration of the different effects between quantum information and classical information for extracting heat from the bath in the QSZE. When the well width $L \rightarrow \infty$ and bath temperature $T \rightarrow \infty$, our results show that the QSZE reduces to the CSZE.

The paper is organized as follows. We introduce our model of the QSZE in section 2. We will present two different cyclic strategies for the QSZE: One with isothermal expansion and the other with adiabatic expansion, and analyze the cyclic processes of the QSZE with fully quantum consideration in section 3. Two

limits of the QSZE at $L \rightarrow \infty$ and $T \rightarrow \infty$ are discussed in section 4. Finally, we present our conclusions in section 5. Remarks on notational details and some technical derivations are given in the appendixes.

2. The model

Consider a single particle of mass m confined to a one-dimensional infinite square well of width L . The eigenvalues E_n and eigenstates $|E_n\rangle$ are

$$E_n(L) = \frac{n^2 \hbar^2 \pi^2}{2mL^2}, \quad n = 1, 2, 3, \dots, \quad (1)$$

$$|E_n(L)\rangle = \begin{cases} \sqrt{\frac{2}{L}} \sin\left[\frac{n\pi(x - L/2)}{L}\right], & n = 2k \\ \sqrt{\frac{2}{L}} \cos\left[\frac{n\pi(x - L/2)}{L}\right], & n = 2k - 1 \end{cases}, \quad (2)$$

where k is a positive integer and $0 \leq x \leq L$.

Assume that the system is initially in thermal equilibrium with a bath at temperature T , the density matrix $\rho_0(L)$ reads as

$$\rho_0(L) = \sum_{n=1}^{\infty} P_n(L) |E_n(L)\rangle \langle E_n(L)|, \quad (3)$$

where $P_n(L) = \frac{e^{-\beta E_n}}{Z(L)}$ is the probability of the particle in the eigenstate $|E_n\rangle$, and satisfies the normalization condition $\sum_{n=1}^{\infty} P_n(L) = 1$. $Z(L) = \sum_{n=1}^{\infty} e^{-\beta E_n}$ is the partition function, $\beta = \frac{1}{k_B T}$ and k_B is the Boltzmann constant. The initial system's internal energy $U_0(L)$ and the initial Von-Neumann entropy S_0 , are given by

$$U_0(L) = \sum_{n=1}^{\infty} P_n(L) E_n(L), \quad (4)$$

$$S_0 = -k_B \text{Tr}(\rho_0 \ln \rho_0) = -k_B \sum_{n=1}^{\infty} P_n(L) \ln P_n(L), \quad (5)$$

respectively.

3. The fully quantum analysis and discussions of the QSZE

In this section we will present two different cyclic strategies, one with isothermal expansion as illustrated in Fig. 2 and the other with adiabatic expansion in Fig. 3. Each strategy consists of four steps: adiabatic insertion, measurement, expansion and removal. The first two steps, adiabatic insertion and measurement, usually are performed simultaneously and can be considered as one. We also assume that measurement is perfect, and the case of imperfect measurement has been discussed in Ref. [24]. In order to reveal the physics behind each process, we calculate the internal energy, work, heat and the entropy change in each step.

Step One: Adiabatic Insertion and Measurement

It is widely accepted that when a wall is inserted at any position of the box, there is no heat and work accompanied by in the CSZE. However, this is not the case in the QSZE. Analogous to the classical insertion, the corresponding quantum process is characterized by increasing the height of the potential

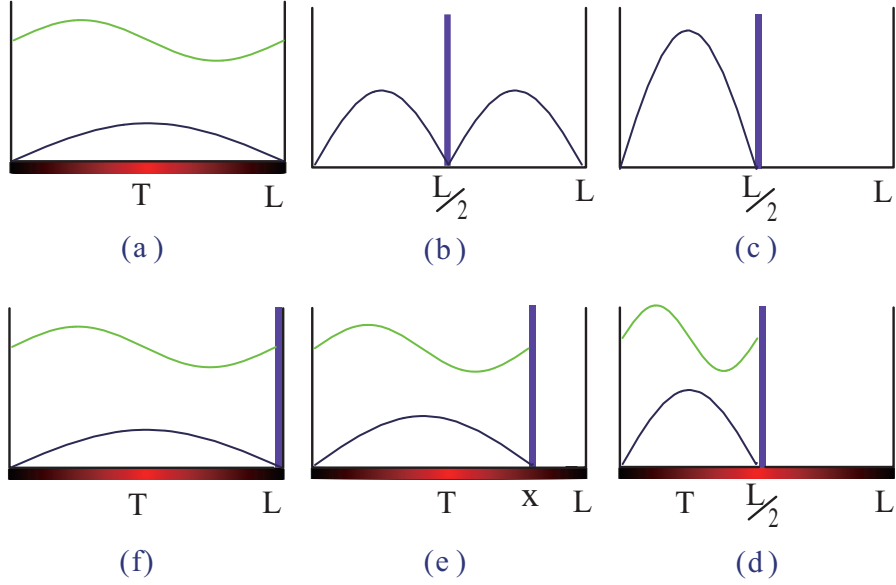


Figure 2: (Color online) Schematic diagram of the thermodynamic processes of the QSZE with an isothermal expansion. For simplicity, we only take two lowest energy levels with odd and even parities as an example to show the rules of energy level redistributions due to the insertion. (a) Initially a single particle is in a thermal equilibrium with a heat bath at temperature T . (b) After adiabatically adding an infinite potential barrier at the center of square well, the well is split into two identical subspaces and before the measurement the particle stays in the left or right subspaces with the same probability. (c) The particle is found in the left subspace after the measurement. (d) The system contacts with the heat bath of temperature T and reaches thermal equilibrium. (e) The system is performed an isothermal expansion. (f) The barrier arrives at the right end of the well and the system returns to its initial state.

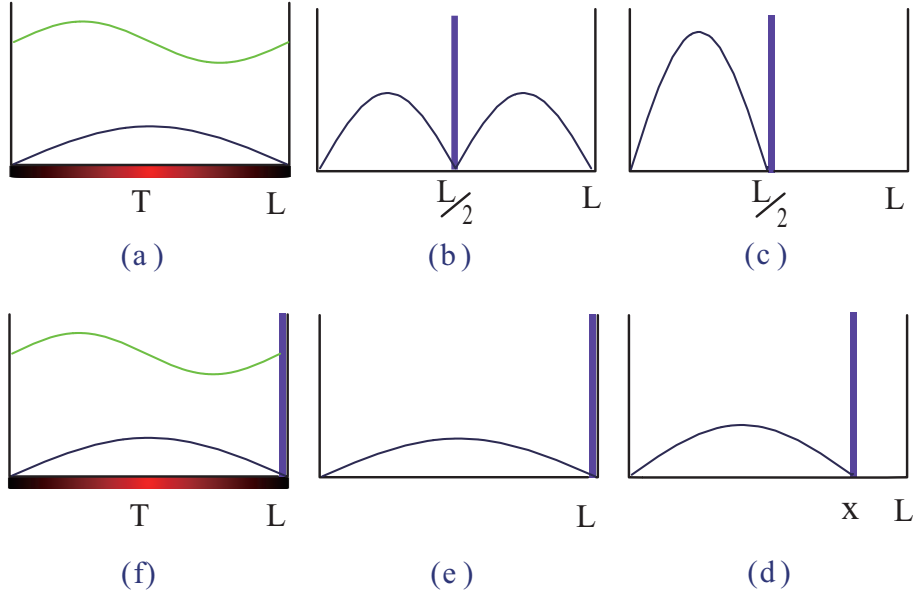


Figure 3: (Color online) Schematic diagram of the thermodynamic processes of the QSZE with an adiabatic expansion. For the same reason as in Fig. 2 we only take two lowest energy levels with odd and even parities as an example. (a) Initially a single particle is in thermal equilibrium with a heat bath at temperature T . (b) After adiabatically adding an infinite potential barrier at the center of square well, the well is split into two identical subspaces and before the measurement the particle stays in the left or right subspaces with the same probability. (c) The particle is found in the left subspace after the measurement. (d) The system is performed an adiabatic expansion. (e) The barrier arrives at the right end of the well. (f) The system contacts with the heat bath and reaches thermal equilibrium.

barrier. Adiabatic insertion $dQ = 0$ implies that the system is isolated from the heat bath and the potential barrier increases very slowly at the center of potential well $x_0 = L/2$. One can model the potential

$$V(x, t) = \begin{cases} \infty, & x < 0, x > L \\ \lambda(t)\delta(x - \frac{L}{2}), & 0 < x < L \end{cases}, \quad (6)$$

where $\lambda(t)$ varies from zero to infinity adiabatically. The insertion of the impenetrable barrier is completed at $\lambda(t) \rightarrow \infty$.

Appendix A shows the detailed calculations of the energy level shifts during the adiabatic insertion (see also [25]). There are two situations when the insertion takes place at $x_0 = L/2$. The *odd-parity* eigenfunctions $|E_{2k}(L)\rangle$ remain the same in full space, because the insertion point is the same as nodes of the odd-parity eigenfunctions such that the system cannot detect and therefore does not resist the insertion. When the insertion is completed, $|E_{2k}(L)\rangle$ become the eigenfunction $|E_k(L/2)\rangle$ in the left and right subspaces with equal probability. Since $E_k(L/2) = E_{2k}(L)$, the energy levels do not shift in this situation.

The insertion changes the *even-parity* eigenfunctions $|E_{2k-1}(L)\rangle$ and the eigenenergies. Interestingly, the eigenenergies also vary with the insertion rate [23]. When the adiabatic insertion is completed, these eigenfunctions become $|E_k(L/2)\rangle$ in the left and right subspaces with equal probability such that $E_k(L/2) = E_{2k}(L)$. This suggests that $E_{2k-1}(L)$ will shift upward to the nearest level $E_{2k}(L)$.

Since the potential tends to infinity when the insertion is completed, the box will be divided into two independent unrelated subspaces, and the particle will be either in the left subspace or the right subspace with the same probability, $P^{(L)} = P^{(R)} = 1/2$. So the cross terms of $\rho^{(L)}$ and $\rho^{(R)}$, in the density matrix of the system ρ_1 , become zero after the insertion and ρ_1 reads as

$$\rho_1 = \frac{1}{2}(\rho^{(L)} + \rho^{(R)}). \quad (7)$$

The density matrices of subspaces $\rho^{(L)}$ and $\rho^{(R)}$ can be expressed as

$$\rho^{(L)} = \sum_{k=1}^{\infty} P_k^{(L)}(\frac{L}{2}) |E_k^{(L)}(\frac{L}{2})\rangle \langle E_k^{(L)}(\frac{L}{2})|, \quad (8)$$

$$\rho^{(R)} = \sum_{k=1}^{\infty} P_k^{(R)}(\frac{L}{2}) |E_k^{(R)}(\frac{L}{2})\rangle \langle E_k^{(R)}(\frac{L}{2})|, \quad (9)$$

and

$$P_k^{(L)}(\frac{L}{2}) = P_k^{(R)}(\frac{L}{2}) = P_{2k}(L) + P_{2k-1}(L), \quad (10)$$

$$E_k^{(L)}(\frac{L}{2}) = E_k^{(R)}(\frac{L}{2}) = E_{2k}(L), \quad (11)$$

where L and R denote the left subspace and the right subspace. The eigenstates $|E_k^{(L)}(L/2)\rangle$ and $|E_k^{(R)}(L/2)\rangle$ correspond to the eigenvalues $E_k^{(L)}(L/2)$ and $E_k^{(R)}(L/2)$ with the well width $L/2$. $P_k^{(L)}(L/2)$ ($P_k^{(R)}(L/2)$) is the probability in the state $|E_k^{(L)}(L/2)\rangle$ ($|E_k^{(R)}(L/2)\rangle$) immediately after the insertion is completed. At this moment the system is not yet in thermal equilibrium with the bath and $P_k^{(L)}(L/2)$ or $P_k^{(R)}(L/2)$ does not satisfy the Boltzmann distribution. The entropy of the system is

$$S_1 = -k_B \text{Tr}(\rho_1 \ln \rho_1) = S_c + h(p), \quad (12)$$

where $S_c = k_B \ln 2$, $h(p) = -k_B \text{Tr}(\rho^{(L)} \ln \rho^{(L)})$ represent the classical information entropy and quantum information entropy of the system respectively (we will sometimes use information to refer to entropy in the following context). The internal energy is

$$U_1 = \frac{1}{2} \sum_{k=1}^{\infty} [P_k^{(L)}(\frac{L}{2}) E_k^{(L)}(\frac{L}{2}) + P_k^{(R)}(\frac{L}{2}) E_k^{(R)}(\frac{L}{2})]. \quad (13)$$

Substitute Eqs. (10) and (11) into Eq. (13) one obtains

$$U_1 = \sum_{k=1}^{\infty} [P_{2k}(L) + P_{2k-1}(L)] E_{2k}(L). \quad (14)$$

The internal energy change merely comes from the work done by the outside agent because the insertion is implemented adiabatically, i.e., $Q_1 = 0$. In general, the measurement is performed without any energy cost [22, 23], so the work W_1 done by the outside agent, in the insertion process, equals the amount of the increased internal energy, ΔU_{10} , that is

$$W_1 = \Delta U_{10} = U_1 - U_0 = \sum_{k=1}^{\infty} P_{2k-1}(L) [E_{2k}(L) - E_{2k-1}(L)]. \quad (15)$$

The total entropy change of the system is

$$\Delta S_{10} = S_1 - S_0 = S_c - [S_0 - h(p)], \quad (16)$$

where $S_0 - h(p) > 0$ and can be easily verified through the inequality $P_{2k-1} \ln P_{2k-1} + P_{2k} \ln P_{2k} < (P_{2k-1} + P_{2k}) \ln (P_{2k-1} + P_{2k})$. Interestingly, Eq. (16) shows that the information change in the insertion consists of two parts: the increased classical information S_c , and the decreased quantum information, $-[S_0 - h(p)]$. This implies that in the insertion process the classical information increases while the quantum information decreases. In addition, it is noted that though the heat exchange is zero in the adiabatic insertion, $dQ = 0$, the entropy change of the system, ΔS_{10} , is not equal to zero because this process is a non-equilibrium process, and the relation $dQ = TdS$ doesn't hold any more.

The classical information and the quantum information are acquired from different origins. The former comes from the position distribution of the particle, while the later comes from the probability distribution of the energy levels in quantum system which is obtained at the expense of the work done by the outside agent. In the whole cycle, they also play different roles. The classical information S_c seems to behave like a feedback controller determining the moving direction of the barrier and does not contribute to the heat absorption, however, the quantum information $S_0 - h(p)$ determines the heat absorbed. We will show the differences specifically in subsequent sections.

We make a measurement to localize the particle in one of two sides of the well. After the measurement the classical information becomes zero and the particle, we assume, is in the left side (same discussions when it is in the right). Then the feedback is finished. Based on the result of the feedback, the barrier will eventually reach the right end of the well. The state ρ_1 , after measurement, collapses into $\rho_2 = \rho^{(L)}$ in the left space. The internal energy of the system now becomes

$$U_2 = \sum_{k=1}^{\infty} P_k^{(L)} \left(\frac{L}{2}\right) E_k^{(L)} \left(\frac{L}{2}\right) = U_1. \quad (17)$$

It is commonly accepted that there is no cost of energy in the measurement process such that the heat absorbed and work done for the system during measurement are zero

$$Q_2 = W_2 = 0. \quad (18)$$

But the entropy of the system changes and becomes

$$S_2 = -k_B \text{Tr}(\rho_2 \ln \rho_2) = -k_B \text{Tr}(\rho^{(L)} \ln \rho^{(L)}) = h(p). \quad (19)$$

The entropy change due to the measurement is $\Delta S_{21} = S_2 - S_1 = -S_c$. This indicates again that we now know exactly the side where the particle is located and the classical information disappears after the feedback.

Step Two: Expansion Process

We will discuss two cyclic strategies associated with different expansions. One is the isothermal expansion described in Fig. 2. The other is that the system first undergoes an adiabatic expansion, and then relaxes to thermal equilibrium by contacting the heat bath as illustrated in Fig. 3.

Case (A) Isothermal Expansion

The isothermal expansion consists of two procedures. We first "hold" the barrier, let the system contact the heat bath and wait until they reach thermal equilibrium, as shown in Fig. 2(c)→(d). There is no work done in this procedure. Second we let the barrier move very slowly and eventually arrive at the right end as shown in Fig. 2(d)→(f). We require that the second procedure be quasi-static such that the system and the heat bath are always in thermal equilibrium. The system state at position x , can be described by the density matrix $\rho(x)$

$$\rho(x) = \sum_{k=1}^{\infty} P_k(x) |E_k(x)\rangle \langle E_k(x)|, \quad (20)$$

where $L/2 \leq x \leq L$, $P_k(x) = \exp[-\beta E_k(x)]/Z(x)$ represents the probability of the particle at k energy level $E_k(x) = \frac{k^2 \hbar^2 \pi^2}{2mx^2}$, and $Z(x) = \sum_{k=1}^{\infty} e^{-\beta E_k(x)}$ is the partition function.

The first procedure, as shown in Fig. 2(d), ends up with the density matrix $\rho_3 = \rho(x = L/2)$,

$$\rho_3 = \sum_{k=1}^{\infty} P_k\left(\frac{L}{2}\right) |E_k\left(\frac{L}{2}\right)\rangle \langle E_k\left(\frac{L}{2}\right)|. \quad (21)$$

The work W_3 and the internal energy U_3 are

$$W_3 = 0, \quad (22)$$

$$U_3 = \sum_{k=1}^{\infty} P_k\left(\frac{L}{2}\right) E_k\left(\frac{L}{2}\right), \quad (23)$$

respectively. The heat absorbed, Q_3 , equals the increase of the internal energy

$$Q_3 = U_3 - U_2 = \sum_{k=1}^{\infty} [P_k\left(\frac{L}{2}\right) - P_k^{(L)}\left(\frac{L}{2}\right)] E_k\left(\frac{L}{2}\right). \quad (24)$$

The entropy of the system is

$$S_3 = -k_B \text{Tr}(\rho_3 \ln \rho_3) = -k_B \sum_{k=1}^{\infty} [P_k\left(\frac{L}{2}\right) \ln P_k\left(\frac{L}{2}\right)]. \quad (25)$$

The entropy change ΔS_{32} in this procedure is

$$\Delta S_{32} = S_3 - S_2 = -k_B [\text{Tr}(\rho_3 \ln \rho_3) - \text{Tr}(\rho_2 \ln \rho_2)]. \quad (26)$$

It shows that the system state changes from ρ_2 to ρ_3 by absorbing heat Q_3 to erase the quantum information ΔS_{32} .

In the second procedure, the system entropy increases gradually and reaches its maximum S_4 at $x = L$ where the system returns to the initial equilibrium state ρ_0 so that

$$\rho_4 = \rho_0, \quad (27)$$

$$U_4 = U_0(L), \quad (28)$$

$$S_4 = S_0. \quad (29)$$

The quantum entropy change is given by

$$\Delta S_{43} = S_4 - S_3 = -k_B [\text{Tr}(\rho_0 \ln \rho_0) - \text{Tr}(\rho_3 \ln \rho_3)]. \quad (30)$$

In the spirit of the Landauer's erasure principle in the CSZE, the classical information is erased gradually with the moving barrier and vanishes when the barrier reaches the end of the well. By contrast the quantum information in the QSZE with the isothermal expansion is erased gradually with the barrier moving until the system returns to the initial state. The absorbed heat Q_4 has erased the quantum information ΔS_{43} . In this way all the quantum information obtained in the insertion and the measurement has been erased completely,

$$S_0 - h(p) - [\Delta S_{32} + \Delta S_{43}] = 0. \quad (31)$$

This indicates that the amount of quantum information obtained equals to the total amount of quantum information erased by absorbing heat in the cycle.

We now find out how much heat is absorbed and how much work is done by the system in the second procedure, by using the first law of thermodynamics $dQ = dU + dW$. Here dQ and dW are the heat absorbed and the work done by the system, respectively [26, 27],

$$dW = - \sum_n P_n dE_n, \quad (32)$$

and

$$dQ = \sum_n E_n dP_n. \quad (33)$$

For slowly moving barrier, we can integrate Eqs. (32) and (33) from $L/2$ to L and obtain the work done by the system and the heat absorbed in the isothermal expansion,

$$W_4 = \sum_{k=1}^{\infty} \int_{L/2}^L P_k(x) dE_k(x) = - \sum_{k=1}^{\infty} \int_{L/2}^L \frac{e^{-\beta E_k(x)}}{Z(x)} dE_k(x) = k_B T \ln \frac{Z(L)}{Z(L/2)}, \quad (34)$$

$$\begin{aligned} Q_4 &= \int_{L/2}^L d[U(x) + W(x)] = U_4 - U_3 + k_B T \ln \frac{Z(L)}{Z(L/2)} \\ &= \sum_{k=1}^{\infty} \{ [P_{2k}(L)E_{2k}(L) + P_{2k-1}(L)E_{2k-1}(L)] - P_k(\frac{L}{2})E_k(\frac{L}{2}) \} + k_B T \ln \frac{Z(L)}{Z(L/2)}. \end{aligned} \quad (35)$$

Here, the absorbed heat Q_4 is exploited to erase the amount of the quantum information ΔS_{43} and brings the system back to the initial thermal equilibrium state ρ_0 from ρ_3 .

The total work W_{exp} and the total heat Q_{exp} in the two procedures are

$$W_{\text{exp}} = W_3 + W_4 = k_B T \ln \frac{Z(L)}{Z(L/2)}, \quad (36)$$

$$Q_{\text{exp}} = Q_3 + Q_4 = - \sum_{k=1}^{\infty} P_{2k-1}(L) [E_{2k}(L) - E_{2k-1}(L)] + k_B T \ln \frac{Z(L)}{Z(L/2)} = -W_1 + W_{\text{exp}}, \quad (37)$$

respectively. Eq. (37) can also be rewritten as $Q_{\text{exp}} + W_1 = W_{\text{exp}}$. It implies that the total heat plus the work done by the external agent in the insertion, $Q_{\text{exp}} + W_1$, equals to the work W_{exp} done by the system in the expansion. Moreover, the quantum information $S_0 - h(p)$ is used to extract heat from the bath and determines how much heat is absorbed or how much work is extracted.

Case (B) Adiabatic Expansion and Thermalization

Similar to the strategy of the isothermal expansion, the process also consists of two procedures. The system first undergoes an adiabatic expansion to the right end of square well, and then contacts the heat bath and relaxes to thermal equilibrium.

The absorbed heat $Q'_3 = 0$ in the adiabatic expansion, as illustrated in Fig. 3(c)→(e). The process is non-equilibrium, and during the process the probability distribution of the energy level is the same as that immediately after the insertion and measurement. Let $P_k(x)$ be the probability at the energy level k and at position x , then $P_k(x) = P_k^{(L)}(\frac{L}{2})$, where $\frac{L}{2} \leq x \leq L$ and $P_k^{(L)}(\frac{L}{2})$ is given in Eq. (10). When the barrier reaches the end of well at $x = L$, the internal energy $U'_3(L)$ can be written as

$$U'_3 = \sum_{k=1}^{\infty} P_k^{(L)}(\frac{L}{2}) E_k(L), \quad (38)$$

and the quantum entropy of the system does not change,

$$S'_3 = S_2 = h(p). \quad (39)$$

Since no heat is absorbed in this process, the work W'_3 equals to the internal energy decrease,

$$W'_3 = U_2 - U'_3 = \sum_{k=1}^{\infty} P_k^{(L)}(\frac{L}{2}) [E_k(\frac{L}{2}) - E_k(L)] = \sum_{k=1}^{\infty} 3[P_{2k}(L) + P_{2k-1}(L)] E_k(L). \quad (40)$$

The system afterwards contacts the heat bath and relaxes to thermal equilibrium (i.e., thermalization), as described in Fig. 3(e)→(f). There is no work done in this process, i.e., $W'_4 = 0$. The system only absorbs heat Q'_4 ,

$$\begin{aligned} Q'_4 &= U_0 - U'_3 \\ &= \sum_{k=1}^{\infty} [P_{2k}(L) E_{2k}(L) + P_{2k-1}(L) E_{2k-1}(L)] - \sum_{k=1}^{\infty} P_k^{(L)}(\frac{L}{2}) E_k(L) \\ &= \sum_{k=1}^{\infty} 3[P_{2k}(L) + P_{2k-1}(L)] E_k(L) - P_{2k-1}(L) [E_{2k}(L) - E_{2k-1}(L)] \\ &= W'_3 - W_1, \end{aligned} \quad (41)$$

and it is clear that Q'_4 is also the total heat absorbed in a cycle with the adiabatic expansion and is exploited to erase the quantum information $\Delta S_0 - h(p)$.

The above analysis shows that the total heat Q'_{exp} and the work W'_{exp} in this strategy are

$$Q'_{\text{exp}} = Q'_3 + Q'_4 = Q'_4, \quad (42)$$

$$W'_{\text{exp}} = W'_3 + W'_4 = W'_3, \quad (43)$$

respectively. It turns out that the heat Q'_{exp} is the same as the heat Q'_4 . Eqs. (41) - (43) demonstrate that the absorbed heat and the work done for the system in this strategy satisfy the relation $Q'_{\text{exp}} + W_1 = W'_{\text{exp}}$. This is the same as that of the isothermal expansion discussed in case (A) (Eq. (37)).

However, it is interesting to note that although the quantum information is the same in both strategies, the results of the work and the heat are different. The heat absorbed and the work done in the QSZE depend on the cyclic strategies, that are deviated from $k_B T \ln 2$ in the CSZE.

Step Three: Removal Process

As mentioned above, the barrier will always end up at the edge of the well when the expansion is completed. Since the system will not be disturbed by removing the barrier, the work and heat are zero, $W_{\text{rem}} = 0$ and $Q_{\text{rem}} = 0$. After the removal the system returns to its initial state. We have analyzed each step of the whole cycle, and now let's check out the relations between the total heat absorbed and the total net work done by the system for each strategy.

For the isothermal case, the total amount of heat Q_{tot} absorbed from the heat bath is

$$Q_{\text{tot}} = Q_1 + Q_2 + Q_3 + Q_4 = Q_{\text{exp}} = - \sum_{k=1}^{\infty} P_{2k-1}(L)[E_{2k}(L) - E_{2k-1}(L)] + k_B T \ln \frac{Z(L)}{Z(\frac{L}{2})}. \quad (44)$$

It shows that Q_{tot} helps to recover the quantum entropy to its maximum value S_0 from $h(p)$. In another word, the quantum information $S_0 - h(p)$ can help the system absorb heat, Q_{tot} , and eventually the system returns to the initial state. The insertion work W_{ins} equals to the minus W_1 , $W_{\text{ins}} = -W_1$, such that the total work W_{tot} and total heat Q_{tot} can be expressed as

$$W_{\text{tot}} = W_{\text{ins}} + W_{\text{exp}} + W_{\text{rem}} = Q_{\text{tot}} = - \sum_{k=1}^{\infty} P_{2k-1}(L)[E_{2k}(L) - E_{2k-1}(L)] + k_B T \ln \frac{Z(L)}{Z(\frac{L}{2})}. \quad (45)$$

It suggests that the total absorbed heat is fully transformed into the effective work of the system, which brings the system into the initial state.

The adiabatic expansion is similar to the isothermal. The total heat Q'_{tot} and total work W'_{tot} satisfy

$$Q'_{\text{tot}} = W'_{\text{tot}} = \sum_{k=1}^{\infty} 3[P_{2k}(L) + P_{2k-1}(L)]E_k(L) - P_{2k-1}(L)[E_{2k}(L) - E_{2k-1}(L)]. \quad (46)$$

Eqs. (45) and (46) show that the total amount of heat absorbed equals to the total work (net work) for both of cyclic strategies.

We now summarize the above discussions as follows:

- (a) We consider the four physical quantities, internal energy, work, heat and entropy (information) in both CSZE and QSZE, and compare their physical properties. In the CSZE the internal energy is conserved during the whole cycle, while in the QSZE the internal energy is changed in the insertion process and in both the isothermal and adiabatic expansion processes. The CSZE only does work in the expansion process during the entire cycle, while in the QSZE work is done in both insertion and expansion processes. In the whole cycle of the CSZE the system only absorbs heat in the expansion process, while in the QSZE it is not that case. In the strategy with the isothermal expansion the heat exchange between the system and the bath occurs twice: After the insertion contacting with the bath and the expansion process, and in the strategy with the adiabatic expansion it occurs only after the expansion contacting the bath. The total absorbed heat is fully exploited to erase the quantum information, while in the CSZE the absorbed heat is only used to erase the classical information. This paper classifies the entropy into the classical entropy and the quantum information entropy. They have different origins. The former reflects the distribution of the particle's position and is the same as that in the classical system. The later is determined by the probability distribution of the energy levels. In the QSZE, they play entirely different roles. The classical information seems to behave like a feedback controller and has no contribution to extracting heat from the bath, while the quantum information acquired during the insertion determines the amount of heat absorbed and work done for the system in a cycle.
- (b) The information itself cannot be converted into energy but it could be exploited to extract work or heat [28] and, in the QSZE, the quantum information determines the amount of heat absorbed and work done by the system.

- (c) Since the insertion and measurement lead to the quantum entropy decrease, the heat must be required, in subsequent processes, to compensate the quantum entropy change and brings the system to the initial state. This is consistent with the spirit of Landauer's information erasing principle [4]. In the original ideal CSEZ the insertion does not need work such that the whole process results in extracting the energy $k_B T \ln 2$ from the bath or the entropy decrease $k_B \ln 2$. On the contrary, the net effect in the QSEZ is that the system absorbs heat from the bath, obtains work in the insertion process, and does work to the outside during the expanding process. It is noted that the work obtained in the insertion process is not the same as that lost in the expansion process. Therefore, although the system returns to the initial state after a cycle, the outside world will not return to its initial state. It turns out that the second law of thermodynamics is not violated in the QSZE.

4. Discussions of two limits

Now let's focus on the two limits $L \rightarrow \infty$ and $T \rightarrow \infty$.

Case (A) $L \rightarrow \infty$:

The partition function $Z(L)$ in the limit $L \rightarrow \infty$ is

$$Z(L) = \sum_{k=1}^{\infty} \exp(-\xi k^2), \quad (47)$$

where $\xi = \frac{1}{k_B T} \frac{\pi^2 \hbar^2}{2mL^2} > 0$. When $L \rightarrow \infty$, the parameter ξ goes to zero and the sum in the above expression can be replaced by an integral

$$Z(L) = \int_1^{\infty} \exp(-\xi x^2) dx \approx \frac{\sqrt{\pi}}{2} \xi^{-\frac{1}{2}} = \frac{L}{\sqrt{\hbar^2/2\pi m k_B T}}, \quad (48)$$

such that

$$\lim_{L \rightarrow \infty} W_{\text{exp}} = \lim_{L \rightarrow \infty} k_B T \ln \frac{Z(L)}{Z(\frac{L}{2})} = k_B T \ln 2. \quad (49)$$

Based on Eqs. (B.5) and (B.10) in Appendix B, we obtain

$$\lim_{L \rightarrow \infty} W_1 = 0, \quad (50)$$

$$\lim_{L \rightarrow \infty} \frac{W_1}{U_0} = 0. \quad (51)$$

The two expressions suggest that the work in the insertion goes to zero when $L \rightarrow \infty$, and it hardly has influence on the system's internal energy. According to Eq. (B.11), we have $\lim_{L \rightarrow \infty} \frac{W_1}{W_{\text{exp}}} = 0$, meaning that W_1 is much smaller than W_{exp} and can be ignored in the total work. We therefore obtain

$$W_{\text{tot}} = W_{\text{ins}} + W_{\text{exp}} + W_{\text{rem}} = k_B T \ln \frac{Z(L)}{Z(\frac{L}{2})} \approx k_B T \ln 2. \quad (52)$$

Eq. (C.8) in Appendix C shows that when $L \rightarrow \infty$, the quantum information $S_0 - h(p)$ acquired in the insertion will reduce to the classical information $k_B \ln 2$. From Eq. (16) the total information change in the insertion will become zero. From above discussions we now arrived at the conclusion: When $L \rightarrow \infty$, none of interested four physical quantities changes in the insertion process. It turns out that the QSZE reduces to the CSZE completely.

Case (B) $T \rightarrow \infty$:

Based on the definition $\xi = \frac{1}{k_B T} \frac{\pi^2 \hbar^2}{2mL^2}$, we can show that $\lim_{T \rightarrow \infty} \xi = \lim_{L \rightarrow \infty} \xi = 0$. Then in high-temperature limit, the partition function $Z(L)$ is the same as Eq. (48). We also have $\lim_{T \rightarrow \infty} W_{\text{exp}} =$

$k_B T \ln 2$ which is the same as that in the $L \rightarrow \infty$ case. Although the inserting work W_1 is divergent [29], Eqs. (B.13, B.14) in Appendix B tell us that when $T \rightarrow \infty$, the internal energy is hardly changed in the insertion because the work by the external agent is much less than the internal energy, such that

$$W_{\text{tot}} = W_{\text{ins}} + W_{\text{exp}} + W_{\text{rem}} \approx k_B T \ln \frac{Z(L)}{Z(\frac{L}{2})} = k_B T \ln 2. \quad (53)$$

Eq. (C.8) in Appendix C shows that the quantum information $S_0 - h(p)$ acquired in the insertion will also reduce to the classical information $k_B \ln 2$ and the total information change in the insertion becomes zero.

From the above discussions it is clear that the QSZE will reduce to the CSZE in $L \rightarrow \infty$ and $T \rightarrow \infty$ limits, that is just the result what we expected.

5. Conclusions

To summarize, we gave the detailed analysis and discussions on the QSZE of a single particle confined to a one-dimensional infinite square well with fully quantum consideration. We for the first time considered the energy level shifts in the insertion, and investigated its effect on physical quantities, such as heat, work, internal energy, and entropy. We found that only the quantum information contributes to the absorbed heat, while the classical information acts like a feedback controller and has no direct effect on the heat absorbed from bath. We also demonstrated that the work done by the system is different from $W_{\text{tot}} = k_B T \ln 2$. It is noted that unlike in the CSZE the external agent in the QSZE has to do some work in the insertion process, and the one does the work in the insertion process is not the same one to which the system does the work in the expansion process. Although the system returns to the initial states after one cycle, the outside world will not return to its initial state. The second law of thermodynamics therefore is not violated in the QSZE. In the limits of $L \rightarrow \infty$ or $T \rightarrow \infty$, the quantum Szilard engine (QSZE) reduces to the classical Szilard engine (CSZE), and the relation $W_{\text{tot}} = k_B T \ln 2$ holds again. Significantly, it is the first demonstration of the different effects between quantum information and classical information for extracting heat from the bath in the QSZE, which provides further understanding of the relationships among heat, information and work from quantum-mechanical perspective.

Acknowledgements

This work is financially supported by National Science Foundation of China (Grants No. 10974016, 11075013, and 11005008), the Natural Science Foundation of Shandong Province, China (Grant No. ZR2011FL009) and the Science and Technology Project of University in Shandong Province, China (Grant No. J12LJ01). L. -A. Wu has been supported by the Ikerbasque Foundation Start-up, the Basque Government (grant IT472-10) and the Spanish MEC (Project No.FIS2009-12773-C02-02).

Appendix A. Derivation of the rule of energy level redistribution in the limit of the height of the barrier tending to infinity

We use $\psi(x)$ instead of $|E(x)\rangle$ to denote the wave function of the system for simplicity. For a single particle of mass m in a potential field described by Eq. (6), $\psi(x) = 0$, in the $x < 0$ and $L < x$ regions. When $0 < x < L$, $\psi(x)$ satisfies Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + \lambda \delta(x - \frac{L}{2}) \psi(x) = E \psi(x), \quad (A.1)$$

and the boundary conditions are

$$\psi(0) = 0, \quad \psi(L) = 0, \quad \psi(\frac{L^+}{2}) = \psi(\frac{L^-}{2}), \quad (A.2)$$

$$\psi'(\frac{L^+}{2}) - \psi'(\frac{L^-}{2}) = \frac{2m\lambda}{\hbar^2} \psi(\frac{L}{2}). \quad (\text{A.3})$$

The general solution of Eq. (A.1) is

$$\begin{cases} \psi_1(x) = A \sin(kx + \varphi_1), & 0 < x < \frac{L}{2} \\ \psi_2(x) = B \sin(kx + \varphi_2), & \frac{L}{2} < x < L \end{cases}, \quad (\text{A.4})$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$. Eq. (A.4) represents the eigenfunctions of the system. We will check that, for the stationary wave functions $\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$, $n = 1, 2, 3, \dots$, when the integer n is even $n = 2, 4, 6, \dots$, the wave function $\psi_n(\frac{L}{2}) = 0$ holds, and all the boundary conditions are satisfied. The solutions of Eq. (A.1) with even n are

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \quad n = 2, 4, 6, \dots \quad (\text{A.5})$$

For the other solutions, we substitute Eq. (A.4) into Eq. (A.3) and have $\varphi_1 = 0$, $\varphi_2 = -kL$, and $B = -A$. Eq. (A.4) can be expressed as

$$\begin{cases} \psi_1(x) = A \sin kx, & 0 < x < \frac{L}{2} \\ \psi_2(x) = A \sin k(L - x), & \frac{L}{2} < x < L \end{cases}. \quad (\text{A.6})$$

Substitute Eq. (A.6) into Eq. (A.3), one obtains

$$-\xi \cot \xi = \frac{mL}{2\hbar^2} \lambda, \quad (\text{A.7})$$

where $\xi = \frac{kL}{2}$. Eq. (A.7) indicates that the function $y = -\xi \cot \xi \geq 0$ is a periodic and monotone increasing function in a period of $\frac{\pi}{2}$. For the arbitrary i_{th} period, the function becomes $y = -\xi_i \cot \xi_i = \frac{mL}{2\hbar^2} \lambda$, where $\xi_i = \frac{k_i L}{2} = \frac{L}{2} \sqrt{\frac{2mE'_i}{\hbar^2}}$, the variable ξ_i satisfies the relation $(i - \frac{1}{2})\pi < \xi_i < i\pi$, and the corresponding value of the function y varies from zero to infinity, namely, the parameter λ changes from zero to infinity continuously. We therefore obtain

$$\frac{(2i-1)^2 \pi^2 \hbar^2}{2mL^2} < E'_i < \frac{(2i)^2 \pi^2 \hbar^2}{2mL^2}. \quad (\text{A.8})$$

Eq. (A.8) can be also expressed as $E_{2i-1} < E'_i < E_{2i}$ which is equivalent to the inequality $(i - \frac{1}{2})\pi < \xi_i < i\pi$. The two sides of the inequality correspond to $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$ respectively. From Eq. (A.5) and Eq. (A.8), we conclude that the even levels E_{2i} don't shift and the odd levels E_{2i-1} shift upwards to E_{2i} when λ goes to infinity.

Appendix B. The work done by the external agent W_1 in two limit cases

Case (A): $L \rightarrow \infty$.

According to Eq. (15), W_1 can be written as

$$W_1 = \frac{\sum_{k=1}^{\infty} P_{2k-1}(L)[E_{2k}(L) - E_{2k-1}(L)]}{\sum_{k=1}^{\infty} P_k} = \frac{\sum_{k=1}^{\infty} \exp[-\xi(2k-1)^2](4k-1)\xi}{\beta \sum_{k=1}^{\infty} \exp(-\xi k^2)}, \quad (\text{B.1})$$

where $\sum_{n=1}^{\infty} P_n(L) = 1$, $P_n(L) = \frac{e^{-\beta E_n}}{Z(L)}$, $Z(L) = \sum_{n=1}^{\infty} e^{-\beta E_n}$ is the partition function and $\xi = \frac{1}{k_B T} \frac{\pi^2 \hbar^2}{2mL^2} > 0$. In view of the following relations:

$$\int_1^{\infty} e^{-\xi x^2} x dx < \sum_{k=1}^{\infty} e^{-\xi k^2} k < \int_0^{\infty} e^{-\xi x^2} x dx, \quad (\text{B.2})$$

and

$$0 < \int_0^1 e^{-\xi x^2} dx < \int_0^1 dx = 1, \quad (\text{B.3})$$

we have

$$\begin{aligned} W_1 &< \frac{1}{\beta} \frac{4 \int_0^{\infty} \xi e^{-\xi x^2} x dx}{\int_1^{\infty} e^{-\xi x^2} dx} \\ &= \xi \frac{1}{\beta} \frac{4 \int_0^{\infty} e^{-\xi x^2} x dx}{\int_0^{\infty} e^{-\xi x^2} dx - \int_0^1 e^{-\xi x^2} dx} \\ &< \xi \frac{1}{\beta} \frac{4 \int_0^{\infty} e^{-\xi x^2} x dx}{\int_0^{\infty} e^{-\xi x^2} dx - \int_0^1 dx} \\ &= \frac{1}{\beta} \frac{2}{\frac{\sqrt{\pi}}{2} \xi^{-\frac{1}{2}} - 1}, \end{aligned} \quad (\text{B.4})$$

where $\beta = \frac{1}{k_B T}$ is a constant.

$$\lim_{L \rightarrow \infty} W_1 = \lim_{\xi \rightarrow 0} W_1 = \lim_{\xi \rightarrow 0} \frac{1}{\beta} \frac{2}{\frac{\sqrt{\pi}}{2} \xi^{-\frac{1}{2}} - 1} = 0. \quad (\text{B.5})$$

In the derivation of Eq. (B.4) we have used the following formula [30]

$$I(n, a) = \int_0^{\infty} e^{-ax^2} x^n dx, \quad (\text{B.6})$$

where $a > 0$, n is an integer and $I(0, a) = \frac{\sqrt{\pi}}{2} a^{-\frac{1}{2}}$, $I(1, a) = \frac{1}{2} a^{-1}$. By using the same formula as Eq. (B.6) and the following relations:

$$\int_1^{\infty} e^{-\xi x^2} x^2 dx < \sum_{k=1}^{\infty} e^{-\xi k^2} k^2 < \int_0^{\infty} e^{-\xi x^2} x^2 dx, \quad (\text{B.7})$$

and

$$0 < \int_0^1 e^{-\xi x^2} x^2 dx < \int_0^1 x^2 dx = \frac{1}{3}, \quad (\text{B.8})$$

we have

$$\begin{aligned} \frac{W_1}{U_0} &= \frac{\sum_{k=1}^{\infty} P_{2k-1}(L) [E_{2k}(L) - E_{2k-1}(L)]}{\sum_{k=1}^{\infty} P_k E_k(L)} \\ &= \frac{\sum_{k=1}^{\infty} \exp[-\xi(2k-1)^2] (4k-1)}{\sum_{k=1}^{\infty} \exp(-\xi k^2) k^2} \\ &< \frac{4 \int_0^{\infty} e^{-\xi x^2} x dx}{\int_1^{\infty} e^{-\xi x^2} x^2 dx} \\ &= \frac{4 \int_0^{\infty} e^{-\xi x^2} x dx}{\int_0^{\infty} e^{-\xi x^2} x^2 dx - \int_0^1 e^{-\xi x^2} x^2 dx} \\ &= \frac{2}{\frac{\sqrt{\pi}}{4} \xi^{-\frac{1}{2}} - \frac{1}{3} \xi}. \end{aligned} \quad (\text{B.9})$$

Thus the limit is

$$\lim_{L \rightarrow \infty} \frac{W_1}{U_0} = \lim_{\xi \rightarrow 0} \frac{W_1}{U_0} = 0. \quad (\text{B.10})$$

From Eqs. (B.5) and (49) we have

$$\lim_{L \rightarrow \infty} \frac{W_1}{W_{\text{exp}}} = 0. \quad (\text{B.11})$$

Case (B): $T \rightarrow \infty$.

Since the limit

$$\lim_{T \rightarrow \infty} \xi = \lim_{L \rightarrow \infty} \xi = 0 \quad (\text{B.12})$$

holds, for $T \rightarrow \infty$, similarly one can obtain the same results as those in the *Case (A)*, i.e.,

$$\lim_{T \rightarrow \infty} \frac{W_1}{U_0} = \lim_{\xi \rightarrow 0} \frac{W_1}{U_0} = 0, \quad (\text{B.13})$$

and

$$\lim_{T \rightarrow \infty} \frac{W_1}{W_{\text{exp}}} = \lim_{\xi \rightarrow 0} \frac{W_1}{W_{\text{exp}}} = 0. \quad (\text{B.14})$$

Appendix C. The entropy change $S_0 - h(p)$ in two limit cases

Here we consider $S_0 - h(p)$ in the two limits $T \rightarrow \infty$ and $L \rightarrow \infty$. From the Eqs. (5,10,19), $\Delta \equiv \frac{S_0 - h(p)}{k_B}$ can be expressed as

$$\begin{aligned} \Delta &= -\text{Tr}(\rho_0 \ln \rho_0) + \text{Tr}(\rho^{(L)} \ln \rho^{(L)}) \\ &= -\sum_{k=1}^{\infty} P_k \ln P_k + \sum_{k=1}^{\infty} (P_{2k-1} + P_{2k}) \ln (P_{2k-1} + P_{2k}) \\ &= -\sum_{k=1}^{\infty} (P_{2k} \ln P_{2k} + P_{2k-1} \ln P_{2k-1}) + \sum_{k=1}^{\infty} (P_{2k-1} + P_{2k}) \ln (P_{2k-1} + P_{2k}) \\ &= \sum_{k=1}^{\infty} P_{2k} \ln \left(1 + \frac{P_{2k-1}}{P_{2k}}\right) + \sum_{k=1}^{\infty} P_{2k-1} \ln \left(1 + \frac{P_{2k}}{P_{2k-1}}\right) \\ &= \frac{\sum_{k=1}^{\infty} e^{-\xi(2k)^2} \ln[1 + e^{\xi(4k-1)}] + \sum_{k=1}^{\infty} e^{-\xi(2k-1)^2} \ln[1 + e^{-\xi(4k-1)}]}{\sum_{k=1}^{\infty} e^{-\xi k^2}}, \end{aligned} \quad (\text{C.1})$$

where $\xi = \frac{1}{k_B T} \frac{\pi^2 \hbar^2}{2mL^2} > 0$, and the above expression satisfies

$$\frac{\sum_{k=1}^{\infty} e^{-\xi k^2} \ln[1 + e^{-\xi(2k+1)}]}{\sum_{k=1}^{\infty} e^{-\xi k^2}} < \Delta < \frac{\sum_{k=1}^{\infty} e^{-\xi k^2} \ln[1 + e^{(-1)^k \xi(2k-1)}]}{\sum_{k=1}^{\infty} e^{-\xi k^2}}. \quad (\text{C.2})$$

For simplicity, let y_L and y_R represent the left side and right side of above inequality. The left side is

$$\begin{aligned}
y_L &= \frac{\sum_{k=1}^{\infty} e^{-\xi k^2} \ln[1 + e^{-\xi(2k+1)}]}{\sum_{k=1}^{\infty} e^{-\xi k^2}} \\
&> \frac{\sum_{k=1}^{\infty} e^{-\xi k^2} \ln[2e^{-\xi(2k+1)}]}{\sum_{k=1}^{\infty} e^{-\xi k^2}} \\
&= \frac{\sum_{k=1}^{\infty} e^{-\xi k^2} \ln 2 - \sum_{k=1}^{\infty} e^{-\xi k^2} \xi(2k+1)}{\sum_{k=1}^{\infty} e^{-\xi k^2}} \\
&= \ln 2 - \frac{2 \sum_{k=1}^{\infty} e^{-\xi k^2} \xi k + \sum_{k=1}^{\infty} e^{-\xi k^2} \xi}{\sum_{k=1}^{\infty} e^{-\xi k^2}} \\
&> \ln 2 - \frac{2 \int_0^{\infty} e^{-\xi x^2} \xi x dx + \int_0^{\infty} e^{-\xi x^2} \xi dx}{\int_0^{\infty} e^{-\xi x^2} dx - \int_0^1 dx} \\
&= \ln 2 - \frac{\frac{\sqrt{\pi}}{2} \xi^{\frac{1}{2}} + 1}{\frac{\sqrt{\pi}}{2} \xi^{\frac{-1}{2}} - 1},
\end{aligned} \tag{C.3}$$

and the limit satisfies

$$\lim_{\xi \rightarrow 0} y_L > \lim_{\xi \rightarrow 0} [\ln 2 - \frac{\frac{\sqrt{\pi}}{2} \xi^{\frac{1}{2}} + 1}{\frac{\sqrt{\pi}}{2} \xi^{\frac{-1}{2}} - 1}] = \ln 2. \tag{C.4}$$

For the right side in Eq. (C.2), we have

$$\begin{aligned}
y_R &= \frac{\sum_{k=1}^{\infty} e^{-\xi k^2} \ln[1 + e^{(-1)^k \xi(2k-1)}]}{\sum_{k=1}^{\infty} e^{-\xi k^2}} \\
&< \frac{\sum_{k=1}^{\infty} e^{-\xi k^2} \ln(2e^{\xi(2k-1)})}{\sum_{k=1}^{\infty} e^{-\xi k^2}} \\
&< \frac{\sum_{k=1}^{\infty} e^{-\xi k^2} \ln 2 + 2 \sum_{k=1}^{\infty} e^{-\xi k^2} \xi k}{\sum_{k=1}^{\infty} e^{-\xi k^2}} \\
&= \ln 2 + \frac{2 \sum_{k=1}^{\infty} e^{-\xi k^2} \xi k}{\sum_{k=1}^{\infty} e^{-\xi k^2}} \\
&< \ln 2 + \frac{2 \int_0^{\infty} e^{-\xi x^2} \xi x dx}{\int_0^{\infty} e^{-\xi x^2} dx - \int_0^1 dx} \\
&= \ln 2 + \frac{1}{\frac{\sqrt{\pi}}{2} \xi^{\frac{-1}{2}} - 1}.
\end{aligned} \tag{C.5}$$

Take the limit $\xi \rightarrow 0$ in Eq. (C.5), we have

$$\lim_{\xi \rightarrow 0} y_R < \lim_{\xi \rightarrow 0} [\ln 2 + \frac{1}{\frac{\sqrt{\pi}}{2} \xi^{\frac{-1}{2}} - 1}] = \ln 2. \tag{C.6}$$

From Eqs. (C4, C6), one obtains

$$\lim_{\xi \rightarrow 0} \Delta = \ln 2. \tag{C.7}$$

According to $\lim_{T \rightarrow \infty} \xi = \lim_{L \rightarrow \infty} \xi = 0$ and $\Delta = \frac{S_0 - h(p)}{k_B}$, we obtain

$$\lim_{L \rightarrow \infty} [S_0 - h(p)] = \lim_{T \rightarrow \infty} [S_0 - h(p)] = \lim_{\xi \rightarrow 0} k_B \Delta = k_B \ln 2. \tag{C.8}$$

References

- [1] H. S. Leff and A. F. Rex, *Maxwell's Demon 2 : Entropy, Classical and Quantum Information, Computing*, Institute of Physics, Bristol, 2003.
- [2] K. Maruyama, F. Nori, and V. Vedral, *Rev. Mod. Phys.* 81 (2009) 1.
- [3] L. Szilard, *Z. Phys.* 53 (1929) 840.
- [4] R. Landauer, *IBM J. Res. Dev.* 5 (1961) 183.
- [5] C. H. Bennett, *Int. J. Theor. Phys.* 21 (1982) 905.
- [6] L. B. Levitin, In: D. Cabile, D. G. Kuper and I. Riess (eds.) *Proc. 13th IUPAP Conf. Stat. Phys.* Hilger, Bristol (1978).
- [7] L. B. Levitin, In: S. Diner, and G. Lochak, (eds.) *Information, Complexity, and Control in Quantum Physics*, pp. 15 – 47. Springer, Berlin (1987).
- [8] L. B. Levitin, In: *Proc. Worksh. on Physics and Computation (PhysComp'92)*, pp. 223 – 226. IEEE Comput. Soc., Los Alamitos (1993).
- [9] M. O. Scully, *Phys. Rev. Lett.* 87 (2001) 220601.
- [10] W. H. Zureck, *Phys. Rev. A* 67 (2003) 012320.
- [11] O. Dahlsten, R. Renner, E. Rieper, and V. Vedral. arXiv:0908.0424.
- [12] M. O. Scully, M. S. Zhubairy, G. S. Agarwal, and H. Walther, *Science* 299 (2003) 862.
- [13] S. W. Kim and M.-S. Choi, *Phys. Rev. Lett.* 95 (2005) 226802.
- [14] S. W. Kim and M.-S. Choi, *J. Kor. Phys. Soc.* 50 (2007) 337.
- [15] M. G. Raizen, A. M. Dudarev, Q. Niu, and N. J. Fisch, *Phys. Rev. Lett.* 94 (2005) 053003.
- [16] R. Marathe and J. M. R. Parrondo, *Phys. Rev. Lett.* 104 (2010) 245704.
- [17] V. Serreli, C.-F. Lee, E. R. Kay, and D. A. Leigh, *Nature* 445 (2007) 523.
- [18] J. J. Thorn, E. A. Schoene, T. Li, and D. A. Steck, *Phys. Rev. Lett.* 100 (2008) 240407.
- [19] G. N. Price, S. T. Bannerman, K. Viering, E. Narevicius, and M. G. Raizen, *Phys. Rev. Lett.* 100 (2008) 093004.
- [20] W. H. Zurek, in *Frontiers of NonEquilibrium Statistical Physics*, edited by G. T. Moore and M.O. Scully, Plenum, New York, 1984.
- [21] S. Lloyd, *Phys. Rev. A* 56 (1997) 3374.
- [22] S. W. Kim, T. Sagawa, S. De Liberato, and M. Ueda, *Phys. Rev. Lett.* 106 (2011) 070401.
- [23] C. M. Bender, D. C. Brody, and B. K. Meister, *Proc. R. Soc. A* 461 (2005) 733.
- [24] T. Sagawa and M. Ueda, *Phys. Rev. Lett.* 102 (2009) 250602.
- [25] Y. D. Zhang, *A Grand Dictionry of Physics Problems And Solutions: Quantum Mechanics*, Science Press, Beijing, 2005.
- [26] T. D. Kieu, *Phys. Rev. Lett.* 93 (2004) 140403; *Eur. Phys. J. D* 39 (2006) 115.
- [27] M. Esposito and S. Mukamel, *Phys. Rev. E* 73 (2006) 046129.
- [28] L. B. Levitin and T. Toffoli, *Theor. Phys.* 10 (2011) 1007.
- [29] H. Dong, D. Z. Xu, C. Y. Cai, and C. P. Sun, *Phys. Rev. E* 83 (2011) 061108.
- [30] R. Baierlein, *Thermal Physics*, Cambridge University Press, Cambridge, UK, 1999.